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Sensitivity Analysis of a Wing Aeroelastic Response

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A scheme is designed and implemented to obtain the sensitivity of the static aeroelastic response of a three-dimensional wing model. The formulation is quite general and accepts any aerodynamics and structural analysis capability. It assumes that for a given shape and elastic deformation, the aerodynamic analysis will provide the pressure distribution on the wing. Similarly, the structural analysis needs to provide only the elastic deformation of the wing, given its shape and pressure distribution. An interface code is written to convert one's analysis output to the other's input, and vice versa. Local sensitivity derivatives are calculated by either analytic methods or finite-difference techniques. A program to combine the local sensitivities, such as the sensitivity of the stiffness matrix or the aerodynamic kernel matrix, into global sensitivity derivatives is developed. The aerodynamic analysis package, FAST, using a lifting surface theory, and a structural package, ELAPS, implementing Giles' equivalent plate model are used. Results with excellent accuracy have been produced for large integrated quantities such as wing tip deflection. The sensitivities of individual generalized displacements and trim angle of attack have been found with fair accuracy. In all cases, accuracy is proportional to the size of the derivatives.

I. Introduction

DURING the design phase of an engineering system, numerous analyses are conducted to predict changes in the characteristics of the system due to changes in design variables. Usually, this process entails perturbing each variable in turn, recalculating the characteristics, and evaluating the sensitivities by a finite-difference calculation. These repeated analyses can drive the cost of design very high. An approach that has found increased interest recently in engineering design is analytical calculation of the sensitivity derivatives.[1] Typically, the analytical approach requires less computational resources than a finite-difference approach, and is less subject to numerical errors (round-off or truncation). The analytical approach is best developed in parallel with the baseline analysis capability since it uses a significant portion of the numerical information generated during that baseline analysis. In the design of modern aircraft, airframe flexibility is a concern from strength, control, and performance standpoints. To properly account for the aerodynamic and structural implications of flexibility, reliable aeroelastic sensitivity analysis is needed. Therefore, both structural and aerodynamic sensitivity analysis capabilities are necessary.

Structural sensitivity analysis methodology has been available for over two decades for both sizing (thickness, cross-sectional properties) and shape (configuration) variables.[2] However, aerodynamic sensitivity analysis has been nonexistent until recently. Some limited aerodynamic sensitivity analysis capability was developed for aircraft in subcritical compressible flow by Hawk and Bristow,[3] but it only handled perturbations in the direction of the thickness of the wing (thickness, camber, or twist distribution). Yates[4] proposed a new approach that considers general geometry variations including planform for subsonic, sonic, and supersonic unsteady, nonplanar lifting-surface theory.

Aeroelastic sensitivity analysis methodology has also been available for more than two decades for structural sizing variables (Haftka and Yates[5]). This is because changes in sizing variables exclusively affect the structural stiffness and mass distribution of the airframe and not its basic geometry. Therefore, structural sensitivity analysis capability is sufficient. However, the lack of development in aerodynamic shape sensitivity analysis explains why there are very few results in aeroelastic shape sensitivity analysis. In a notable exception, Haftka et al.[6] designed a sailplane wing under aeroelastic constraints and analyzed the design model with vortex lattice and finite-element methods. A finite-difference aeroelastic sensitivity analysis capability is made possible by 1) devising a reduced order model to describe the wing static aeroelastic response, and 2) using exact perturbation analysis to approximate changes in the vorticity vector with changes in the geometry.

Barthelemy and Bergen[7] demonstrated the feasibility of analytically calculating the sensitivity of wing static aeroelastic characteristics to changes in wing shape. Of interest also was the fact that the curvature of the aeroelastic characteristics was small enough that analytical sensitivity derivatives could be used to approximate them without costly reanalyses for large perturbations of the design variables.

The dynamic aeroelastic phenomena is also of interest to designers and it would be advantageous to the aircraft designers to have a tool that can be used to predict the changes in flutter speed with the changes in basic shape parameters.

As is the case for static aeroelastic response, sensitivity calculations have only been available for structural sizing parameters. For examples, Rudisill and Bhatia[8] developed expressions for the analytical derivatives of the eigenvalues, reduced frequency, and flutter speed with respect to structural parameters for use in minimizing the total mass. However, this method is limited because the structural parameters are sizing variables such as cross-sectional areas, plate thickness, and diameters of spars.

Pedersen and Seyranian[9] examined the change in flutter load as a function of change in stiffness, mass, boundary conditions, or load distribution. They showed how sensitivity

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analysis can be performed without any new eigenvalue analysis. The solution to the main and adjoint problem provide all the necessary information for evaluating sensitivities. Their paper mainly focuses on column and beam critical load distributions.

In a recent study, Kapania et al.[10] obtained the sensitivity of a wing flutter response to changes in its geometry. Specifically, the objective was to determine the derivatives of flutter speed and frequency with respect to wing area, aspect ratio, taper ratio, and sweep angle. The study used the equivalent plate model of Giles[11,12] to represent the wing structure. The aerodynamic loads were obtained using the modified strip analysis of Yates[13] to analyze flutter characteristics for finite-span swept and unswept wings. It is noted that Yates' modified strip theory was used quite recently by Landsberger and Dugundji,[14] with a modification for chamber effects given by Spielberg,[15] to study the flutter and divergence of a composite plate.

An excellent examination of various issues involved in aeroelastic analysis was recently published by Borland.[16] The study compares and contrasts "integrated" aeroelastic systems vs "interfaced" systems.

This article deals with determining the sensitivity of the various static aeroelastic responses to the variations in various shape parameters, namely 1) wing area, 2) sweep, 3) aspect ratio, and 4) the taper ratio. The aeroelastic responses are the generalized aeroelastic displacements and the trim angle of attack. The sensitivities have been obtained by differentiating the constitutive equations. It is shown that the resulting sensitivity equations can be reformulated into a variation of the Sobieski's global sensitivity equations[17] (GSE) approach. Both schemes give the various global sensitivities (i.e., the sensitivity including all interdisciplinary interactions) in terms of local sensitivities (i.e., the sensitivities obtained at the discipline level). A key feature that distinguishes this study from the study by Barthelemy and Bergen[10] is the use of a more realistic aerodynamic model, FAST,[18] that uses a lifting surface theory as opposed to a lifting line theory employed in the earlier study. The formulation is designed to be quite general so that it is applicable with any aerodynamic code which, for a given geometry and structural deformations, provides aerodynamic pressures on the wing surface. To facilitate the calculation of the shape sensitivities of various quantities (required in aeroelastic analyses), the pressure distribution is first represented as a double series in Chebyshev polynomials. The displacements of the wing are obtained using an iterative scheme. To validate this more general formulation, sensitivity of the static aeroelastic response of an example wing has been obtained. The results are compared with those obtained by using a purely finite-difference approach. A good agreement is obtained.

II. Sensitivity Equations

A variation of Sobieski's GSE was developed. A variety of local sensitivity data is combined to produce global sensitivity results. Here, the term local sensitivity refers to the sensitivity of an item within a particular discipline, such as the sensitivity of the wing stiffness matrix to a change in wing sweep. Global sensitivities are dependent on the interaction of the disciplines. A global sensitivity example is the variation of wing deflected shape with respect to a change in wing taper ratio.

Anticipating the availability of nonlinear aerodynamic models in the future, the formulation does not assume a linear dependence between the lift generated and the generalized coordinates, and the initial angle of attack. This needs an iterative process to calculate the trim angle of attack to produce required lift.

The governing equations of motion for the aeroelastic analysis and the lift can be written as

$$[K]\{C\} = \{Q\} \tag{1}$$

$$\frac{nW}{2} = \int\int_{\Omega} p(x, y)\, d\Omega \tag{2}$$

where $[K]$ is the stiffness matrix, $\{C\}$ is the vector of unknown generalized displacements, $\{Q\}$ is the vector of generalized forces, n is the load factor, W is the aircraft weight, $p(x, y)$ is the wing pressure field, and Ω is the wing surface area.

The vector of generalized forces can be obtained as

$$Q_i = \int\int_{\Omega} p(x, y)\gamma_i(x, y)\, dx\, dy \tag{3}$$

where $\gamma_i(x, y)$ is the ith nondimensional displacement function used in the displacement model:

$$W(x, y) = \sum_{i=1}^{np} \gamma_i(x, y)C_i \tag{4}$$

These γ satisfy the geometric boundary conditions for a cantilever plate. The C are the generalized displacements.

To facilitate both the integration and subsequent sensitivity calculations, a coordinate transformation was used to simplify the integration limits. This was accomplished using the following transformation:

$$x(\eta, \xi) = \sum_{j=1}^{4} N_j(\eta, \xi)x_j \tag{5}$$

$$y(\eta, \xi) = \sum_{j=1}^{4} N_j(\eta, \xi)y_j \tag{6}$$

where $N_j(\eta, \xi)$ are the shape functions, and x_j and y_j are the coordinates of the four corner points of the wing. The shape functions are given as

$$N_i(\eta, \xi) = (1 + \xi\xi_i)(1 + \eta\eta_i)/4 \tag{7}$$

where η_i and ξ_i are the coordinates of the node i in the $\eta - \xi$ system. Note that this transformation will change the domain of the wing to a square $(-1 \leq \eta \leq 1; -1 \leq \xi \leq 1)$.

As a first step to obtain the generalized forces, the pressure distribution on the wing was represented as a series of global interpolation functions. This can be represented in a generic form as

$$p(\eta, \xi) = \sum_{j=1}^{M} \beta^j(\eta, \xi)a^j \tag{8}$$

where a^j can be considered as the generalized pressure coefficients, and $\beta^j(\eta, \xi)$ are some known interpolation functions of η and ξ. A large number of interpolating polynomials are available in the literature.[19,20] In this study, a tensor product of Chebyshev polynomials is used. They were chosen for their ability to accurately fit a curve with a small number of terms, due to their orthogonality properties. The pressure distribution used in this study can therefore be written

$$p(\eta, \xi) = \sum_{q=0}^{Q} \sum_{p=0}^{P} a_{pq}T_p(\eta)T_q(\xi) \tag{9}$$

Here T_p is the Chebyshev polynomial of order p. A comparison of Eqs. (8) and (9) shows for instance that $\beta^1 = T_0(\eta)T_0(\xi)$ and $a^1 = a_{00}$. The aerodynamic load coefficients a_{pq} can be easily obtained if the values of the pressure coefficients at the zeros of the Chebyshev polynomials are known.

The integral for a generalized force Q_i then becomes

$$Q_i = \int_{-1}^{1}\int_{-1}^{1} p(\eta, \xi)\gamma_i(\eta, \xi)|J(\eta, \xi)|\, d\eta\, d\xi \tag{10}$$

where $|J(\eta, \xi)|$ is the Jacobian of the coordinate transformation. The generalized force Q_i can be written as

$$Q_i = \sum_{j=1}^{M} A_{ij} a^j \tag{11}$$

In matrix form

$$\{Q\} = [A]\{a\} \tag{12}$$

where a typical term A_{ij} is given as

$$A_{ij} = \int_{-1}^{1} \int_{-1}^{1} \beta^j(\eta, \xi) \gamma_i(\eta, \xi) |J(\eta, \xi)| \, \mathrm{d}\eta \, \mathrm{d}\xi \tag{13}$$

Similarly, the lift equation can be written as

$$\frac{nW}{2} = \sum_{j=1}^{M} a^j L^j = L^T a \tag{14}$$

where

$$L^j = \int_{-1}^{1} \int_{-1}^{1} \beta^j(\eta, \xi) |J(\eta, \xi)| \, \mathrm{d}\eta \, \mathrm{d}\xi \tag{15}$$

A Gauss integration scheme was used to perform the double integrals in Eq. (13) whereas the L^j were obtained exactly.

A. Aeroelastic Response

The aeroelastic response was obtained in an iterative fashion. In that, the pressure distribution on the wing is first obtained by assuming the wing to be rigid and having an angle of attack of 1 deg (throughout the span). The pressure distribution thus obtained is used to obtain the vector of generalized forces [Eq. (12)] which in turn is used to obtain the vector of generalized displacements [Eq. (1)]. The elastic displacements are superimposed on the rigid wing and a new pressure distribution on the wing is obtained. This pressure distribution is then used to obtain the generalized displacements. The total lift on the wing is calculated, and a new trim angle of attack is obtained by dividing the total required lift by the current calculated lift, and multiplying by the current trim angle of attack. This process is repeated until a converged value of the trim angle of attack is achieved for the wing. No relaxation is necessary to achieve convergence for the cases studied.

B. Sensitivity Analysis

The goal of this analysis is to produce values for the global sensitivities $\mathrm{d}C/\mathrm{d}r_l$ and $\mathrm{d}\alpha/\mathrm{d}r_l$. Equations (1), (12), and (14) can be used to perform the shape sensitivity analysis of static aeroelastic response. In the following development $\partial(\ )/\partial(\ )$ indicates a local, single discipline term, and $\mathrm{d}(\ )/\mathrm{d}(\ )$ indicates a global or total derivative. Taking derivatives of the equilibrium and the trim equation, with respect to the shape variable r_l (namely sweep, aspect ratio, wing area, taper ratio), we obtain

$$[K]\left\{\frac{\mathrm{d}C}{\mathrm{d}r_l}\right\} + \left[\frac{\mathrm{d}K}{\mathrm{d}r_l}\right]\{C\} = \left\{\frac{\mathrm{d}Q}{\mathrm{d}r_l}\right\} \tag{16}$$

$$\frac{\mathrm{d}(nW/2)}{\mathrm{d}r_l} = \left\{\frac{\partial L}{\partial r_l}\right\}^T \{a\} + \{L\}^T \left\{\frac{\mathrm{d}a}{\mathrm{d}r_l}\right\} = 0 \tag{17}$$

Note that the derivative of lift is zero, because we require the total lift acting on the wing to remain constant. Additionally, L is purely a function of geometry, so its partial derivative is the same as its total derivative.

The vector $\{\mathrm{d}Q/\mathrm{d}r_l\}$ can be obtained as

$$\frac{\mathrm{d}Q_i}{\mathrm{d}r_l} = \sum_{j=1}^{M} \left(\left[\frac{\partial A_{ij}}{\partial r_l}\right] a^j + A_{ij} \frac{\mathrm{d}a^j}{\mathrm{d}r_l} \right) \tag{18}$$

where

$$\frac{\mathrm{d}a^j}{\mathrm{d}r_l} = \frac{\partial a^j}{\partial r_l} + \sum_{n=1}^{np} \left(\frac{\partial a^j}{\partial C_n} \frac{\mathrm{d}C_n}{\mathrm{d}r_l} \right) + \frac{\partial a^j}{\partial \alpha} \frac{\mathrm{d}\alpha}{\mathrm{d}r_l} \tag{19}$$

and where $\partial a^j/\partial r_l$ is the local sensitivity of the aerodynamic generalized pressure coefficients and can be obtained while performing the aerodynamic analysis; $\partial a^j/\partial C_n$ is the derivative of the generalized pressure coefficient with respect to a generalized displacement C_n, and $\mathrm{d}\alpha/\mathrm{d}r_l$ is the derivative of the trim angle of attack with respect to r_l. Note that the matrix $[A]$ is also purely a function of geometry; thus its partial and total derivative are the same.

In matrix form, the global sensitivity of generalized forces becomes

$$\left\{\frac{\mathrm{d}Q}{\mathrm{d}r_l}\right\} = \left[\frac{\partial A}{\partial r_l}\right]\{a\} + [A]\left\{\frac{\partial a}{\partial r_l}\right\} + [A]\left[\frac{\partial\{a\}}{\partial\{C\}}\right]\left\{\frac{\mathrm{d}C}{\mathrm{d}r_l}\right\} + \frac{\mathrm{d}\alpha}{\mathrm{d}r_l}[A]\left\{\frac{\partial a}{\partial \alpha}\right\} \tag{20}$$

It is noted that the major computational expense is the determination of $\partial\{a\}/\partial\{C\}$.

The sensitivity of generalized displacements, therefore, becomes [combining Eqs. (16) and (20)]

$$\left[[K] - [A]\left[\frac{\partial\{a\}}{\partial\{C\}}\right]\right]\left\{\frac{\mathrm{d}C}{\mathrm{d}r_l}\right\} = \left[\frac{\partial A}{\partial r_l}\right]\{a\} + [A]\left\{\frac{\partial a}{\partial r_l}\right\} + \frac{\mathrm{d}\alpha}{\mathrm{d}r_l}[A]\left\{\frac{\partial a}{\partial \alpha}\right\} - \left[\frac{\mathrm{d}K}{\mathrm{d}r_l}\right]\{C\} \tag{21}$$

In this equation, all the terms on the right side are known except for $\mathrm{d}\alpha/\mathrm{d}r_l$. This can be obtained by considering the sensitivity of the lift equation [Eq. (14)]. This equation can be written as

$$\left\{\frac{\partial L}{\partial r_l}\right\}^T \{a\} + \{L\}^T \left\{\frac{\partial a}{\partial r_l}\right\} + \{L\}^T \left[\frac{\partial\{a\}}{\partial\{C\}}\right]\left\{\frac{\mathrm{d}C}{\mathrm{d}r_l}\right\} + \frac{\mathrm{d}\alpha}{\mathrm{d}r_l}\{L\}^T\left\{\frac{\mathrm{d}a}{\mathrm{d}\alpha}\right\} = 0 \tag{22}$$

The required sensitivity derivatives can be then obtained by simultaneously solving the sets of Eqs. (21) and (22). The terms $[K]$, $[A]$, $\{a\}$, $\{C\}$, and $\{L\}$ are all quantities known from the converged baseline configuration. A finite-difference technique is used to calculate the terms, $\mathrm{d}\{a\}/\mathrm{d}\alpha$, $\mathrm{d}[K]/\mathrm{d}r_l$, and $\partial\{a\}/\partial r_l$. The terms $\partial[A]/\partial r_l$ and $\partial\{L\}/\partial r_l$ are computed analytically. The terms $\mathrm{d}\{C\}/\mathrm{d}r_l$ and $\mathrm{d}\alpha/\mathrm{d}r_l$ are the unknowns found by solving this system of equations. With a little rearranging Eqs. (21) and (22) become

$$\begin{bmatrix} [K] - [A]\left[\dfrac{\partial\{a\}}{\partial\{C\}}\right] & -[A]\left\{\dfrac{\partial\{a\}}{\partial\alpha}\right\} \\ \{L\}^T\left[\dfrac{\partial\{a\}}{\partial\{C\}}\right] & \{L\}^T\left\{\dfrac{\partial\{a\}}{\partial\alpha}\right\} \end{bmatrix} \begin{Bmatrix} \dfrac{\mathrm{d}\{C\}}{\mathrm{d}r_l} \\ \dfrac{\mathrm{d}\alpha}{\mathrm{d}r_l} \end{Bmatrix} = \begin{Bmatrix} \left[\dfrac{\partial A}{\partial r_l}\right]\{a\} + [A]\left\{\dfrac{\partial a}{\partial r_l}\right\} - \left[\dfrac{\partial K}{\partial r_l}\right]\{C\} \\ -\left\{\dfrac{\partial L}{\partial r_l}\right\}^T\{a\} - \{L\}^T\left\{\dfrac{\partial a}{\partial r_l}\right\} \end{Bmatrix} \tag{23}$$

Generating this left side "sensitivity" matrix by finite differences is extremely computationally expensive. The term

$\partial\{a\}/\partial\{C\}$ is particularly expensive. It requires calling the aerodynamic code one time for each element in the $\{C\}$ vector. The sensitivity matrix is valid for a particular base geometry, regardless of which r_i is of interest. It is only generated once and saved for future uses. The left side sensitivity matrix may be used with a different right side to determine the converged wing aeroelastic response. This equation is derived in Appendix B:

$$\begin{bmatrix} [K] - [A]\left[\dfrac{\partial\{a\}}{\partial\{C\}}\right] & -[A]\left\{\dfrac{\partial\{a\}}{\partial\alpha}\right\} \\ \{L\}^T\left[\dfrac{\partial\{a\}}{\partial\{C\}}\right] & \{L\}^T\left\{\dfrac{\partial\{a\}}{\partial\alpha}\right\} \end{bmatrix} \begin{Bmatrix} \{C\} \\ \alpha \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ nW/2 \end{Bmatrix} \tag{24}$$

If a sensitivity calculation is to be performed as well, then this approach in determining the aeroelastic response is more efficient. If only the converged wing configuration is needed, the iterative approach is slightly better. The iterative approach is also more general; it can be directly applied to nonlinear aerodynamic codes, whereas Eq. (24) is only applicable for linear aerodynamics.

III. Implementation

The combination of realistic aerodynamics and structural models in a modular manner with shape sensitivity code requires a systematic approach. A scheme of calling the aerodynamic and structural codes to produce a converged static wing loading and shape was developed. In addition a set of "neutral format" data files were defined. Here, a neutral format data file is a file that contains certain data at certain spots, regardless of the package that originally generated the data. This scheme makes the replacement of analysis packages practical and relatively simple. These files include the base geometry and initial deflection values, the intermediate pressure loading and structural deflections, and the final converged wing loading and deflection.

The aeroelastic problem is broken into subproblems (or blocks) by discipline. The aerodynamic and structural blocks are called iteratively to produce a converged static wing loading and shape. Shape sensitivity values for this converged wing are then obtained. These values could then be used in an optimization scheme to modify the baseline geometry.

Each block operates completely independent of the other. It reads from several neutral format input files, performs its calculations, and generates one or more neutral format output files. Thus, any aerodynamic and structural analysis capability may be used. Only new input and output "adapter" programs need be written to add a new analysis package to the system. These two adapter programs must convert the neutral format data files to and from the new package's native format.

A. Aerodynamics

The aerodynamic block is responsible for generating the loads on the wing. It reads as input the wing geometry parameters and the current wing deflections. It is able to output the pressure on the wing at arbitrary points. Currently, the aerodynamic analysis is being performed by the program FAST. This lifting panel code was developed at NASA, Langley. It is based on the theory developed by Yates, and was implemented by Desmarais and Bennett.[18] Originally developed for a CDC Cyber computer, it has been ported to the VAX/VMS and IBM/CMS operating systems. Adapter programs to convert to and from native FAST data files to the defined neutral format were also developed.

B. Structures

The structural block is responsible for calculating the deflection of the wing. It is given the wing geometry and wing loading. It calculates the deflected shape of the wing. Currently, Giles' ELAPS[12] code is being used to perform the structural analysis. This Ritz method program was developed at NASA, Langley. It has been adapted for use on both the VAX and IBM systems. Adapter programs have been developed to convert wing pressures to ELAPS generalized forces and to convert its deflection outputs to neutral form.

C. Converged Static Wing Configuration

The aerodynamic and structural blocks have been combined to produce a converged static wing configuration. The two sections are called iteratively to produce a wing shape and loading that are mutually consistent for a particular flight condition. This iterative technique is used in order to keep this model more general. If, in the future, it is desired to use nonlinear aerodynamic or structural codes, it can be done with little effort.

An alternative technique has also been developed to compute the trim angle of attack and wing generalized deflections. This technique involves significantly more setup effort, but gives the converged displacement and angle of attack in one step. It should be noted that only the linearity of the two current analysis programs makes this one-step solution possible. The equations involved are discussed in Appendix B.

IV. Results

This research has produced a variety of useful results. The aeroelastic code does an excellent job of calculating the converged wing loading and deflections for a particular flight condition. The sensitivity derivative calculations do an excellent job of predicting shape sensitivities. Difficulty has been encountered in the exact calculation of derivatives with very small values, but work is continuing in this area. The numerical results were obtained for the wing shown in Fig. 1. It was structurally modeled using a box beam model detailed in Ref. 10. The material properties were $E_{11} = E_{22} = 6.89 \times 10^{10}$, $G_{12} = 2.65 \times 10^{10}$, and $\nu_{12} = 0.3$.

The behavior of the aerodynamic coefficients $\{a\}$ that are produced by FAST are of interest because several perturbed runs of the program are needed to produce finite difference derivatives for later use by the sensitivity program. As Figs. 2–3 show, the overall trend of these aerodynamic coefficients is smooth, but with significant "wiggle" in these curves. This wiggle makes accurate finite differencing problematical on the affected a. These curves are the worst cases, many of the other curves are much smoother.

To overcome this problem, a higher-order finite-difference scheme using a large step size was used. This scheme

$$f'(x) \approx (1/12h)[f(x - 2\Delta x) - 8f(x - \Delta x) + 0f(x) + 8f(x + \Delta x) - f(x + 2\Delta x)] \tag{25}$$

has an accuracy of $O(\Delta x^4)$.

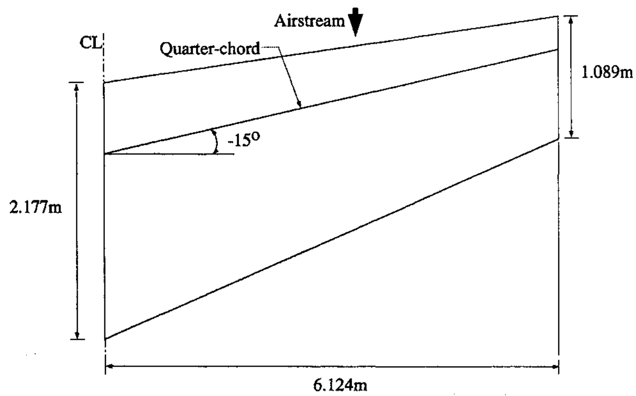

Fig. 1 Baseline wing configuration.

Note, these figures are for single elements of the sixty-element $\{a\}$ vector. These graphs can be used to estimate the best step size for a single element, but not for the entire vector. In an attempt to discover an optimal step size to maximize the accuracy of the vector as a whole, the L-2 norm of the difference between two finite differences was minimized. In this case, for each of a wide variety of step sizes, a forward and central first derivative finite difference was calculated with the central difference used as a reference. The logarithm of the average of the square of the difference of these two values was plotted. The step size that produced the minimum difference was chosen as our optimal step size. Note, this is not in any way a percentage error, but instead the average absolute squared error. Figure 4 shows the variation of this pseudoerror with step size in α and aspect ratio. Similar curves were generated for the other three r_i.

The individual deflection coefficients (C) show some wiggle, similar to the aerodynamic terms, but the effect is much smaller. Since these derivatives are only used to confirm the sensitivity results, a step size study was less important. Nevertheless, one was performed to insure that these derivatives would be as accurate as possible.

Derivatives of the stiffness matrix are also calculated by finite difference for use in the sensitivity code. A step size study identical to the aerodynamic code study was undertaken. While the term "L-2 norm" is not strictly accurate for a matrix operation, the process was the same. The difference between each term of a central- and a forward-differenced derivative was squared and summed. The behavior of this pseudoerror measurement is also plotted in Fig. 4. The optimal step size was used by a central difference in the actual sensitivity calculation.

The variation of the trim angle of attack with respect to the wing area is shown in Fig. 5. The solid line shows the converged results from the iterative aerodynamic and structures combination. The various dashed lines show the variation predicted by the sensitivity derivatives at the different base configurations. The prediction goes through the converged value at the base geometry and is linear with a slope equal to the sensitivity derivative. The desired result is for this line to be tangent to the converged data curve.

Similarly, the sensitivity of the trim angle of attack to changes in the wing aspect ratio is shown in Fig. 6. The solid line shows the converged iterative results, and the dashed lines show the predicted variation by having a slope equal to the calculated sensitivity derivative.

Figures 7 and 8 show the converged and predicted values for the angle-of-attack variation with respect to taper ratio and sweep. It is obvious from Fig. 7 that the obtained value of the sensitivity of the angle of attack with respect to the taper ratio is not very accurate corresponding to the taper ratio values of 0.5 and 0.8. However, note that the value of the converged angle of attack is almost insensitive to the variation in taper ratio at those values of taper ratio. The inaccuracy in the present results can be attributed to the nu-

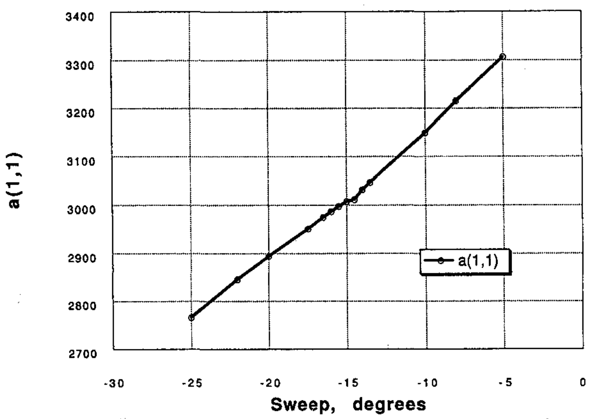

Fig. 2 Pressure expansion coefficient $a(1, 1)$ vs sweep.

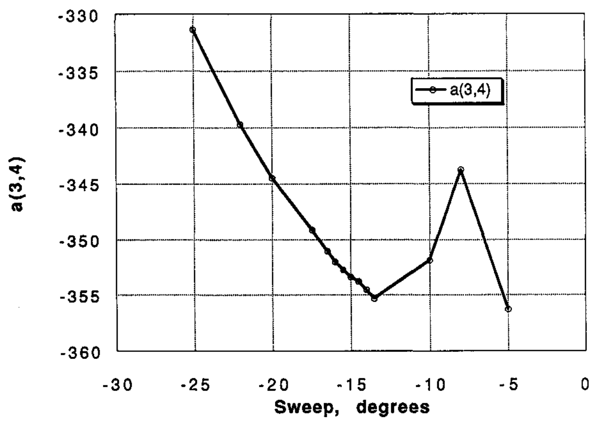

Fig. 3 Pressure expansion coefficient $a(3, 4)$ vs sweep.

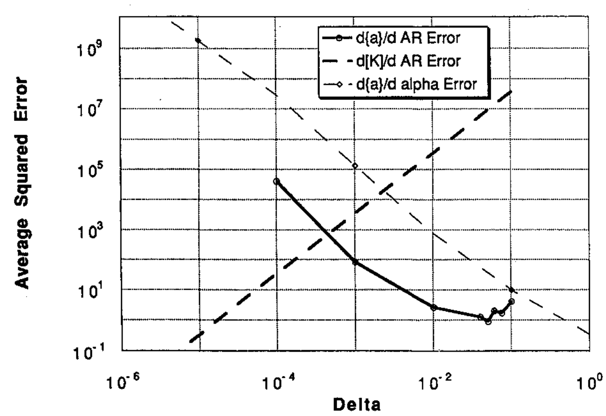

Fig. 4 Finite-difference pseudoerrors.

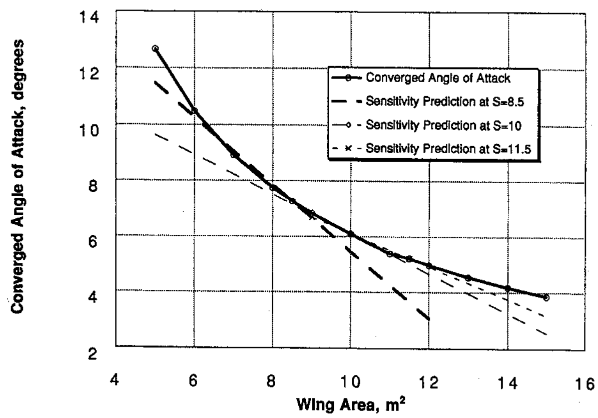

Fig. 5 Trim angle of attack vs wing area.

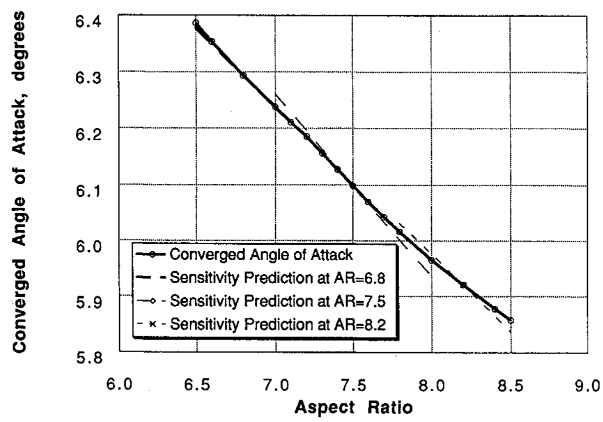

Fig. 6 Trim angle of attack vs aspect ratio.

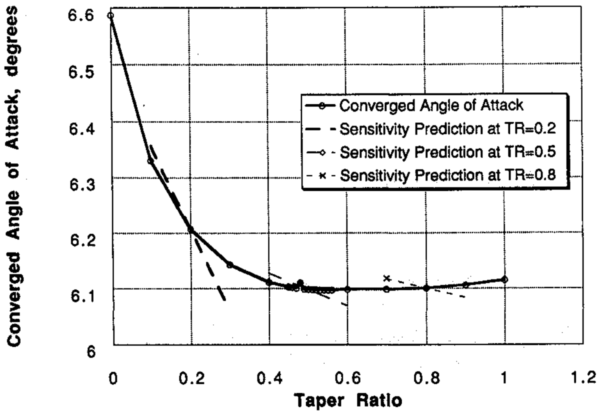

Fig. 7 Trim angle of attack vs taper ratio.

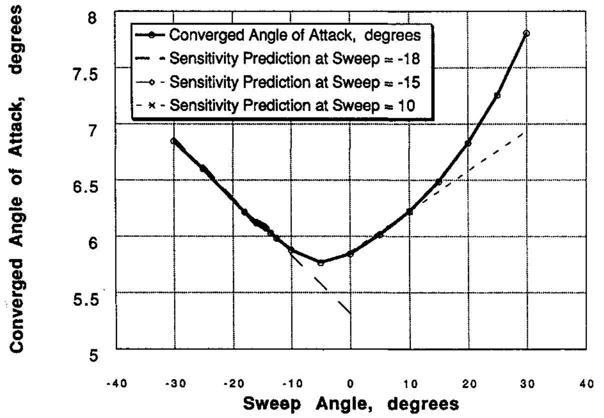

Fig. 8 Trim angle of attack vs sweep.

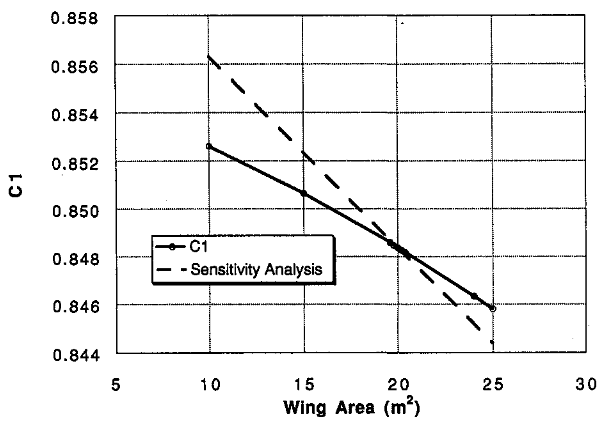

Fig. 9 Deflection coefficient C1 vs area.

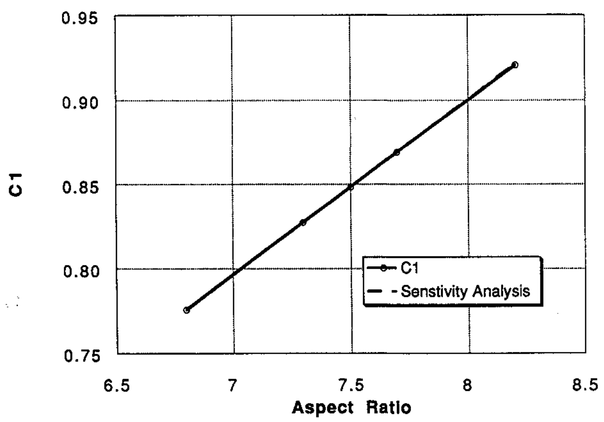

Fig. 10 Deflection expansion coefficient C1 vs aspect ratio.

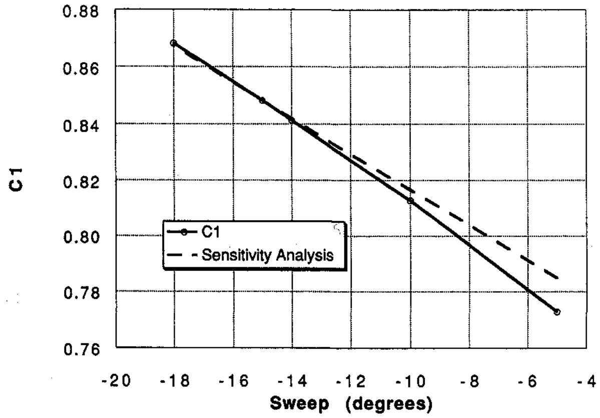

Fig. 11 Deflection expansion coefficient C1 vs sweep.

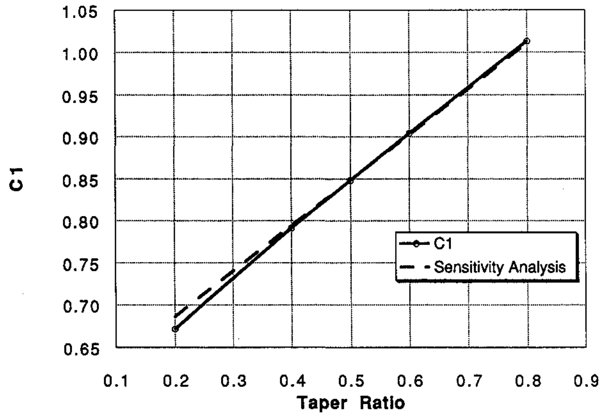

Fig. 12 Deflection expansion coefficient C1 vs taper ratio.

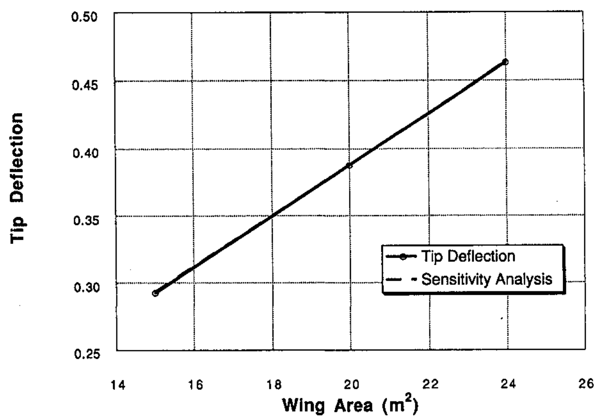

Fig. 13 Wing tip deflection vs area.

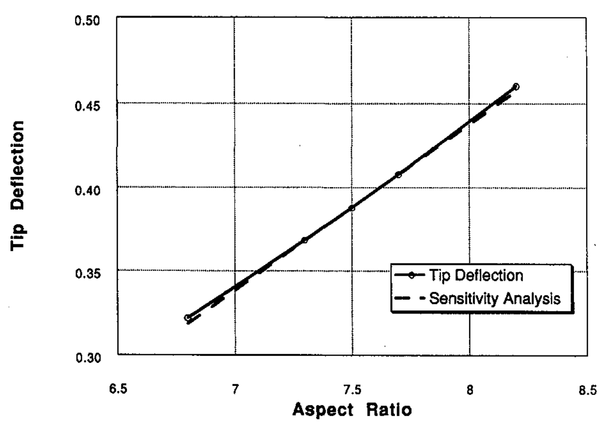

Fig. 14 Wing tip deflection vs aspect ratio.

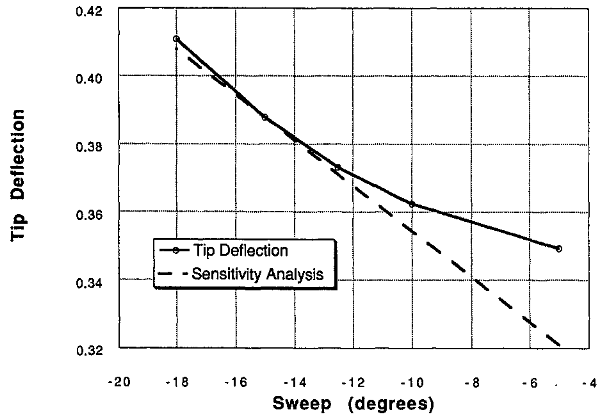

Fig. 15 Wing tip deflection vs sweep.

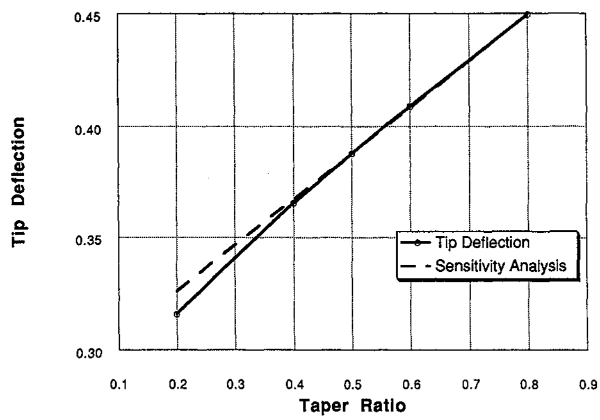

Fig. 16 Wing tip deflection vs taper ratio.

Table 1 Comparison of finite difference and analytic logarithmic derivatives

Term	Finite difference	Analytic	% Error
$\frac{d\alpha}{dS}$	-1.188194	-1.129775	4.917
$\frac{d\alpha}{dAR}$	-0.366320	-0.463287	29.906
$\frac{d\alpha}{d\lambda}$	-1.85613×10^{-2}	-2.40882×10^{-2}	29.777
$\frac{d\alpha}{d\Lambda}$	0.098611	0.126455	28.236
$\frac{dc_1}{dS}$	-5.83136×10^{-4}	-9.77324×10^{-3}	40.333
$\frac{dc_1}{dAR}$	0.122537	0.123591	0.8581
$\frac{dc_1}{d\lambda}$	0.332564	0.329906	-3.957
$\frac{dc_1}{d\Lambda}$	0.122932	0.112007	-8.887
$\frac{dTipDef}{dS}$	0.984423	0.978191	0.2812
$\frac{dTipDef}{dAR}$	1.888806	1.916328	1.457
$\frac{dTipDef}{d\lambda}$	0.273598	0.265756	2.866
$\frac{dTipDef}{d\Lambda}$	0.262560	0.258709	1.467

merical problems associated with determining derivatives that are almost zero.

Also of interest are deflection sensitivities. Figures 9–12 show the predicted and actual variation of a few of the deflection expansion coefficients. Perhaps more important is the sensitivity of the deflections themselves. Figures 13–16 show the sensitivity of the leading-edge tip deflection. The sensitivity of the coefficient C_1 with respect to wing area is not very accurate (see Fig. 9) as C_1 is almost insensitive to the variation in the wing area. The inaccuracies obtained in calculating this sensitivity do not, however, affect the sensitivity of the tip deflection with respect to wing area. This can be observed in Fig. 13. From Figs. 13–16 it is clear that the present formulation yields very accurate shape sensitivities for the aeroelastic tip deflections and can be used in optimization studies. The coefficient errors largely disappear in the integrated quantities.

These errors are largely numerical in origin. Variables with very small logarithmic derivatives will be difficult to differentiate numerically regardless of the scheme used.[21] Table 1 shows a variety of variables, their logarithmic derivatives, and the error in calculating them. As can be seen from the table, some of the intermediate results with very low logarithmic derivatives show as much as a 40% error. However, the final integrated tip deflection results show a maximum of a 3% error.

V. Conclusions

In this research, a variation of Sobieski's GSE is implemented to obtain the global sensitivity of the static aeroelastic responses. The scheme is independent of the analysis code used to obtain aerodynamic data.

The results show good accuracy for integrated quantities such as tip displacements, but less accuracy for individual displacement coefficients or trim angle of attack. In general, the accuracy decreases noticeably when the size of the derivative decreases.

Appendix A: Comparison Between the Present Formulation and Sobieski's GSE

In Ref. 17, Sobieski presented two different formulations (GSE1 and GSE2) to obtain the global sensitivities of a multidisciplinary system in terms of the sensitivities of the subsystems, called local sensitivities. It is of interest to compare the present formulation [Eq. (23)] with Sobieski's GSE.

In terms of the Sobieski's first formulation, called GSE1, the governing equations for the wing can be written as

Trim

$$f_\alpha = L^T(\{r_l\}, \alpha)\{a\} - (nW/2) = 0 \tag{A1}$$

Aerodynamics

$$\{f_a(\alpha, \{a\}, \{C\})\} = \{0\} \tag{A2}$$

Structures

$$\{f_C\} = [K(r_l)]\{C\} - [A(r_l)]\{a\} = \{0\} \tag{A3}$$

Then the GSEs are

$$\begin{bmatrix} \frac{\partial f_\alpha}{\partial \alpha} & \frac{\partial f_\alpha}{\partial \{a\}} & \frac{\partial f_\alpha}{\partial \{a\}} \\ \frac{\partial \{f_a\}}{\partial \alpha} & \frac{\partial \{f_a\}}{\partial \{a\}} & \frac{\partial \{f_a\}}{\partial \{a\}} \\ \frac{\partial \{f_C\}}{\partial \alpha} & \frac{\partial \{f_C\}}{\partial \{a\}} & \frac{\partial \{f_C\}}{\partial \{a\}} \end{bmatrix} \begin{Bmatrix} \frac{d\alpha}{dr_l} \\ \frac{d\{a\}}{dr_l} \\ \frac{d\{C\}}{dr_l} \end{Bmatrix} = - \begin{Bmatrix} \frac{\partial f_\alpha}{\partial r_l} \\ \frac{\partial \{f_a\}}{\partial r_l} \\ \frac{\partial \{f_C\}}{\partial r_l} \end{Bmatrix} \tag{A4}$$

or

$$\begin{bmatrix} 0 & \{L\}^T & 0 \\ \dfrac{\partial\{f_a\}}{\partial\alpha} & \dfrac{\partial\{f_a\}}{\partial\{a\}} & \dfrac{\partial\{f_a\}}{\partial\{a\}} \\ 0 & -[A] & [K] \end{bmatrix} \begin{Bmatrix} \dfrac{d_\alpha}{dr_l} \\ \dfrac{d\{a\}}{dr_l} \\ \dfrac{d\{C\}}{dr_l} \end{Bmatrix} = -\begin{Bmatrix} \dfrac{\partial}{\partial r_l}\{L\}^T\{a\} \\ \dfrac{\partial\{f_a\}}{\partial r_l} \\ \dfrac{\partial}{\partial r_l}[K]\{C\} - \dfrac{\partial}{\partial r_l}[A]\{a\} \end{Bmatrix} \tag{A5}$$

Note that, since we did not have access to $\{f_a\}$ directly, we could not use the above form of the Sobieski's GSE.

In terms of Sobieski's GSE2 formulation, the governing equations for the system at hand are

Trim

$$\alpha = \alpha(\{a\}, r_l) \tag{A6}$$

Aerodynamics

$$\{a\} = \{a(\alpha, \{C\}, r_l)\} \tag{A7}$$

Structures

$$\{C\} = [K(r_l)]^{-1}[A(r_l)]\{a\} \tag{A8}$$

The global sensitivity equations are

$$\begin{bmatrix} I & -\dfrac{\partial\alpha}{\partial\{a\}} & -\dfrac{\partial\alpha}{\partial\{C\}} \\ -\dfrac{\partial\{a\}}{\partial\alpha} & I & -\dfrac{\partial\{a\}}{\partial\{C\}} \\ -\dfrac{\partial\{C\}}{\partial\alpha} & -\dfrac{\partial\{C\}}{\partial\{a\}} & I \end{bmatrix} \begin{Bmatrix} \dfrac{d\alpha}{dr_l} \\ \dfrac{d\{a\}}{dr_l} \\ \dfrac{d\{C\}}{dr_l} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial\alpha}{\partial r_l} \\ \dfrac{\partial\{a\}}{\partial r_l} \\ \dfrac{\partial\{C\}}{\partial r_l} \end{Bmatrix} \tag{A9}$$

Using Eqs. (A6–A8), the above set of equations becomes

$$\begin{bmatrix} I & -\dfrac{\partial\alpha}{\partial\{a\}} & 0 \\ -\dfrac{\partial\{a\}}{\partial\alpha} & I & -\dfrac{\partial\{a\}}{\partial\{C\}} \\ 0 & -[K]^{-1}[A] & I \end{bmatrix} \begin{Bmatrix} \dfrac{d\alpha}{dr_l} \\ \dfrac{d\{a\}}{dr_l} \\ \dfrac{d\{C\}}{dr_l} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \dfrac{\partial\{a\}}{\partial r_l} \\ \dfrac{\partial}{\partial r_l}[[K]^{-1}[A]]\{a\} \end{Bmatrix} \tag{A10}$$

We could not use this formulation, since $\partial\alpha/\partial\{a\}$ cannot be easily found. If we take the first equation from (A5) and the last two equations from (A10), we can write a system as follows:

$$\begin{Bmatrix} \dfrac{\partial\{a\}}{\partial r_l} \\ \dfrac{\partial\{C\}}{\partial r_l} \\ \left\{\dfrac{\partial L}{\partial r_l}\right\}^T\{a\} \end{Bmatrix} = \begin{bmatrix} [I] & -\left[\dfrac{\partial\{a\}}{\partial\{C\}}\right] & -\dfrac{d\{a\}}{d\alpha} \\ -[K]^{-1}[A] & [I] & [0] \\ -\{L\}^T & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dfrac{d\{a\}}{dr_l} \\ \dfrac{d\{C\}}{dr_l} \\ \dfrac{d\alpha}{dr_l} \end{Bmatrix} \tag{A11}$$

For the current study, the global derivatives of $\{a\}$ aren't necessary, so eliminating the first line of Eq. (A11) and rearranging it will give us Eq. (23). Thus, our formulation is a mix of GSE1 and GSE2.

Appendix B: Simultaneous Determination of $\{C\}$ and α

For the case of linear aerodynamics, one is not required to use an iterative scheme to obtain the aeroelastic solution. For such a case the vector of aerodynamic pressures p_i at some discrete points can be written as

$$\{p\} = [\bar{A}]\{C\} + \alpha\left\{\frac{\partial p}{\partial \alpha}\right\} \tag{B1}$$

where

$$\bar{A}_{ij} = \frac{\partial p_i}{\partial C_j}$$

The generalized aerodynamic coefficients can be written as

$$\{a\} = [R]\{p\} \tag{B2}$$

where R is an interpolation matrix that converts discrete p to our generalized a. For our case, $\{p\}$ is taken at Chebychev points. Equation (B1) is substituted into Eq. (B2). Then, the equilibrium equations for the structure are updated from Eqs. (1) and (12):

$$[K]\{C\} = [A][R][\bar{A}]\{C\} + \alpha[A][R]\left\{\frac{\partial p}{\partial \alpha}\right\} \tag{B3}$$

The trim equation is adapted from Eq. (14).

$$n\frac{W}{2} = \{L\}^T[R][\bar{A}]\{C\} + \alpha\{L\}^T[R]\left\{\frac{\partial p}{\partial \alpha}\right\} \tag{B4}$$

Note that for our case

$$[R][\bar{A}] = \frac{\partial\{a\}}{\partial\{C\}}$$

The governing equations, for the case of linear aerodynamics, are

$$\begin{bmatrix} [K] - [A]\dfrac{\partial\{a\}}{\partial\{C\}} & [A]\dfrac{\partial\{a\}}{\partial\alpha} \\ \{L\}^T\left[\dfrac{\partial\{a\}}{\partial\{C\}}\right] & \{L\}^T\dfrac{\partial\{a\}}{\partial\alpha} \end{bmatrix} \begin{Bmatrix} C \\ \alpha \end{Bmatrix} = \begin{Bmatrix} 0 \\ \dfrac{nW}{2} \end{Bmatrix} \tag{B5}$$

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